# Behaviour of Thin Walled Stiffened Plates under In-Plane Axial Loads

**B.Tech Project Report**

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# Certificate

It is certified that the work contained in this report titled "Behaviour of Thin Walled Stiffened Plates under In-Plane Axial Loads" is the original work done by Abhijeet Singh (Y4008) & Manish Kumar (Y4210), and has been carried out under my supervision.

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# Abstract

Unlike other slender structural members like columns, plates have the capability to resist significant amount of compressive axial load after they start to buckle. Advantage can be taken of this special characteristic of plates while determining the allowable load-deflection criteria for the purpose of design of structures. Some established approximate methods of analysis have been used in this project to analyze orthotropic plates for post buckling behaviour. The approach can be easily extended to the analysis of structures where the properties are different in two orthogonal directions because of provision of stiffeners.

Firstly effective width of a stiffened plate has been found out considering it to be an orthotropic plate and then the result has been compared with the codal provisions. Then expression for critical load for buckling of a stiffened plate has been derived.

Post buckling strength of orthotropic has been considered and expression for effective width of the loading in post buckled state has been found out by two methods: 1) Stress Method 2) Successive approximation method. Load versus deflection curve has been drawn to see the buckling mode and post buckling behavior of the plate. Then comparison has been done between the effective width found out by the two methods. Finally a conclusion has been drawn based on results and graphs obtained.

# Acknowledgement

We are immensely grateful to our guide, Dr. Ashwini Kumar for the guidance he provided to us during this project. Dr. Kumar has always been ready to help us in any difficulty we faced and explained the correct approach to the problem. He has not only helped us by giving extremely helpful comments and suggestions regarding the work but also by providing the required material to study and have a better understanding of the subject.

We would also like to thank all the faculty and staff members of the Department of Civil Engineering, IIT Kanpur who made us skilled enough to work on this project by providing a highly fruitful training at this institute.

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# Introduction

## 1.1 IS Code Provisions

IS: 801-1975 suggests an expression to calculate the effective thickness of a stiffened plate. If the intermediate stiffeners are spaced so closely that the flat width ratio between stiffeners does not exceed $(w/t)_{lim}$ , all the stiffeners may considered effective. Only for the purposes of calculating the flat width ratio of entire multiple-stiffened element, such element shall be considered as replaced by an element without intermediate stiffeners whose w is the whole width between webs or from wed to edge stiffener, and whose equivalent thickness $h_{eff}$ is determined as follows:

$$h_{eff} = \sqrt[3]{\frac{12 I_s}{w}} \qquad (1.1)$$

Where $I_s$ = moment of inertia of the full area of the multiple stiffened element, including the intermediate stiffeners, about its own centroidal axis

$(w/t)_{lim}$ is given by

“Maximum allowable overall flat width ratio w/h disregarding intermediate stiffeners and taking h as the actual thickness of the element shall be as follows:

For stiffened compression element with both longitudinal edges connected to other stiffened elements= 500”

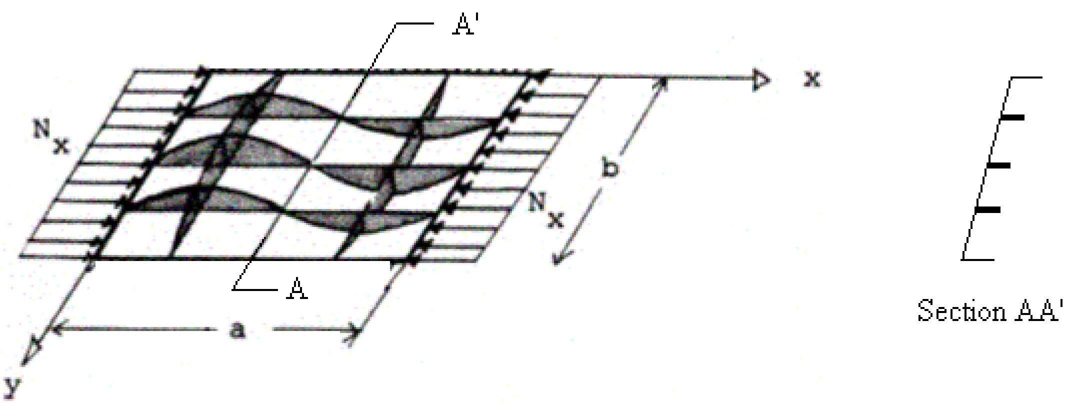


**Fig 1.1:** Buckled shape

A closer examination of the behavior of the element in compression reveals that the given formula by code needs to be reviewed. Introduction of Intermediate stiffeners transfer the plate into an orthotropic plate having much higher flexural rigidities. Plate is being analyzed considering different flexural rigidities in two directions.

## 1.2 Developing the differential equation

The equilibrium equation governing the buckling of a thin plate is given by

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0 \quad (1.2)$$

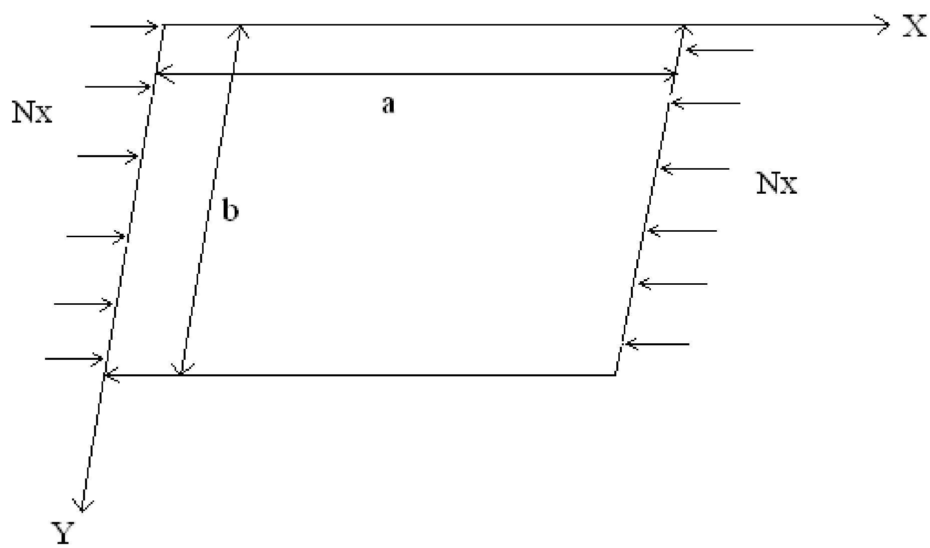


**Fig: 1.2:** Rectangular plate under axial load

We have the following relationships

$$\sigma_{xx} = \frac{E_x}{1-\nu_x\nu_y}[\varepsilon_x + \nu_y\varepsilon_y] \quad (1.3a)$$

$$\sigma_{yy} = \frac{E_y}{1-\nu_x\nu_y}[\varepsilon_y + \nu_x\varepsilon_x] \quad (1.3b)$$

$$\sigma_{xy} = G_{xy}\gamma_{xy} \quad (1.3c)$$

The strains $\varepsilon_x$, $\varepsilon_y$ and $\gamma_{xy}$ can be expressed in terms of displacements as

$$u = -z\frac{\partial w}{\partial x} \qquad v = -z\frac{\partial w}{\partial y} \tag{1.4a}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{1.4b}$$

Substituting the values of u and v in strain equations and then strain values in stress train relationships we get $\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}$ (1.5a)

Therefore

$$\sigma_{xx} = -\frac{E_x}{1-\nu_x\nu_y} z[\frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2}] \tag{1.6a}$$

$$\sigma_{yy} = \frac{E_y}{1-\nu_x\nu_y} z[\frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2}] \tag{1.6b}$$

$$\tau_{xy} = -2G_{xy} z \frac{\partial^2 w}{\partial x \partial y} \tag{1.6c}$$

Consider rectangular plate axially compressed in one direction and simply supported along all the edges.

For this case of plate having thickness “h” and given dimensions, we have

$$M_x = \int_{-h/2}^{+h/2} \sigma_{xx} z dz$$

$$= \frac{E_x}{12(1-\nu_x\nu_y)} h^3 z \left[\frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2}\right] \tag{1.7}$$

or

$$M_x = -D_x\left[\frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2}\right] \tag{1.8a}$$

$$M_y = -D_y\left[\frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2}\right] \tag{1.8b}$$

Similarly

$$M_{xy} = \int_{-h/2}^{+h/2} \sigma_{xy} z dz$$

$$= -\frac{1}{6} G_{xy} h^3 \frac{\partial^2 w}{\partial xy} \tag{1.8c}$$

or

$$M_{xy} = -D_{xy} \frac{\partial^2 w}{\partial xy} \tag{1.8d}$$

where
$$D_x = \frac{E_x h3}{12(1-\nu_x \nu_y)} \tag{1.9a}$$

$$D_y = \frac{E_y h3}{12(1-\nu_x \nu_y)} \tag{1.9b}$$

$$D_{xy} = \frac{G_{xy}}{6} h^3 \tag{1.9c}$$

are called flexural rigidities.

Now substituting the values of moments in governing differential equations

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} = -D_x \frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2}\right]$$

$$\frac{\partial^2 M_x}{\partial x^2} = -D_x\left[\frac{\partial^4 w}{\partial x^4} + \nu_y \frac{\partial^4 w}{\partial x^2 \partial y^2}\right] \tag{1.10a}$$

Similarly

$$\frac{\partial^2 M_y}{\partial y^2} = -D_y \left[\frac{\partial^4 w}{\partial y^4} + \nu_x \frac{\partial^4 w}{\partial x^2 y^2}\right] \tag{1.10b}$$

And

$$\frac{\partial^2 M_{xy}}{\partial x \partial y} = -D_{xy} \frac{\partial^4 w}{\partial x^2 y^2} \tag{1.10c}$$

Substituting for all the expressions

$$-D_x \left[\frac{\partial^4 w}{\partial x^4} + \nu_y \frac{\partial^4 w}{\partial x^2 \partial y^2}\right] - 2D_{xy} \frac{\partial^4 w}{\partial x^2 y^2} - D_y \left[\frac{\partial^4 w}{\partial y^4} + \nu_x \frac{\partial^4 w}{\partial x^2 y^2}\right] + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0$$

or

$$N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = D_x \left[\frac{\partial^4 w}{\partial x^4} + \nu_y \frac{\partial^4 w}{\partial x^2 \partial y^2}\right] + 2D_{xy} \frac{\partial^4 w}{\partial x^2 y^2} + D_y \left[\frac{\partial^4 w}{\partial y^4} + \nu_x \frac{\partial^4 w}{\partial x^2 y^2}\right] \tag{1.11}$$

In this case we are considering rectangular plate axially compressed in one direction and simply supported along all the edges.

**Boundary Conditions:**

$w = 0$ for x = 0, a (1.12a)

$w = 0$ for y = 0, b (1.12b)

$M_x = 0$, $M_y = 0$ at edges

Hence

$$\frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2} = 0 \text{ for x= 0, a} \tag{1.13a}$$

$$\frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2} = 0 \text{ for y= 0, b} \tag{1.13b}$$

Also since curvature of plate will be zero at the boundaries,

$$\frac{\partial^2 w}{\partial y^2} = 0 \text{ for } x = 0, a \tag{1.14a}$$

$$\frac{\partial^2 w}{\partial x^2} = 0 \text{ for } y = 0, b \tag{1.14b}$$

Substituting from (1.14a) in (1.13a) & from (1.14b) in (1.13b)

$$\frac{\partial^2 w}{\partial x^2} = 0 \text{ for } x = 0, a \tag{1.15a}$$

$$\frac{\partial^2 w}{\partial y^2} = 0 \text{ for } y = 0, b \tag{1.15b}$$

Hence from all the conditions derived, it can be said that all the four edges of the plate will be undeflected as well as curvature along an axis perpendicular to edge will be zero at the edge.

As $N_y$ and $N_{xy}$ are zero, the final equation can also be written as

$$D_x\left[\frac{\partial^4 w}{\partial x^4} + \nu_y \frac{\partial^4 w}{\partial x^2 \partial y^2}\right] + 2D_{xy}\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y\left[\frac{\partial^4 w}{\partial y^4} + \nu_x \frac{\partial^4 w}{\partial x^2 \partial y^2}\right] + N_x \frac{\partial^2 w}{\partial x^2} = 0 \tag{1.16}$$

Let the solution be

$$w(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} Sin\frac{m\pi x}{a} Sin\frac{n\pi y}{b} \quad \text{for m = 1,2,3…..}\infty \text{ ; n = 1,2,3…….}\infty \tag{1.17}$$

Substituting in our governing differential equation

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\left[\begin{matrix} D_x\left(m^4\pi^4 + \nu_y\lambda^2 m^2 n^2 \pi^4\right) + \\ D_y\left(\lambda^4 n^4 \pi^4 + \nu_x \lambda^2 m^2 n^2 \pi^4\right) + 2D_{xy}\lambda^2 m^2 n^2 \pi^4 - \mu^2 m^2 \pi^2 \end{matrix}\right] Sin\frac{m\pi x}{a} Sin\frac{n\pi y}{b} = 0 \tag{1.18}$$

where $\mu^2 = a^2 N_x$ and aspect ratio of plate $\lambda = \frac{a}{b}$

As $A_{mn} = 0$ gives trivial solution, $A_{mn} \neq 0$ gives

$$D_x\left(m^4\pi^4 + \nu_y\lambda^2 m^2 n^2 \pi^4\right) + D_y\left(\lambda^4 n^4 \pi^4 + \nu_x\lambda^2 m^2 n^2 \pi^4\right) + 2D_{xy}\lambda^2 m^2 n^2 \pi^4 - \mu^2 m^2 \pi^2 = 0 \tag{1.19}$$

or

$$N_x = D_x \frac{\pi^2}{a^2}\left(m^2 + \nu_y \lambda^2 n^2\right) + D_y \frac{\pi^2}{a^2}\left(\frac{\lambda^4 n^4}{m^2} + \nu_x \lambda^2 n^2\right) + 2\frac{D_{xy}}{a^2}\pi^2 \lambda^2 n^2 \tag{1.20}$$

As n increases, $N_x$ increases. Hence for the lowest value of $N_x$, n must assume one as the numerical value. This implies that the plate buckles with one half sine wave along the y-direction. The number of half sine waves along x-direction that correspond to minimum value of $N_x$ can be obtained by taking derivative of $N_x$ w.r.t. m, with n = 1.

$$\frac{\partial N_x}{\partial m} = 0 \quad => \quad D_x(2m) + D_y\left(-\frac{2\lambda^4}{m^3}\right) = 0$$

$$\Rightarrow \quad m = \frac{a}{b}\sqrt[4]{\frac{D_y}{D_x}} \tag{1.21}$$

Substituting the value of m,

$$\left(N_x\right)_{cr} = \frac{\pi^2}{b^2}\left[2\sqrt{D_x D_y} + \left(D_x \nu_y + D_y \nu_x\right) + 2D_{xy}\right] \tag{1.22}$$

For non-integer values of $\lambda$, the buckling load is higher than that for integer values. For such cases, equation (1.20) can be written as

$$N_x = D_x \frac{\pi^2}{b^2}\psi \tag{1.23}$$

where $\psi$ =buckling load coefficient

After putting the value of $n$=1, $\psi$ can be written as

$$\psi = \left[\frac{m^2}{\lambda^2} + \frac{c\lambda^2}{m^2} + 2c\nu_x + \frac{4c\left(1 - c\nu_x^2\right)}{\left(1 + c + 2c\nu_x\right)}\right] \tag{1.24}$$

where $c = \frac{E_y}{E_x} = \frac{\nu_y}{\nu_x}$ = ratio of properties in two orthogonal directions.

Hence the buckling load coefficient $\psi$ can be plotted for integer values of m and the minimum value for each aspect ratio can be taken as the governing buckling load coefficient.

Here min value of $\psi$ is plotted against aspect ratio $\lambda$ for $\nu_x = 0.25$ and $c = 0.2$.

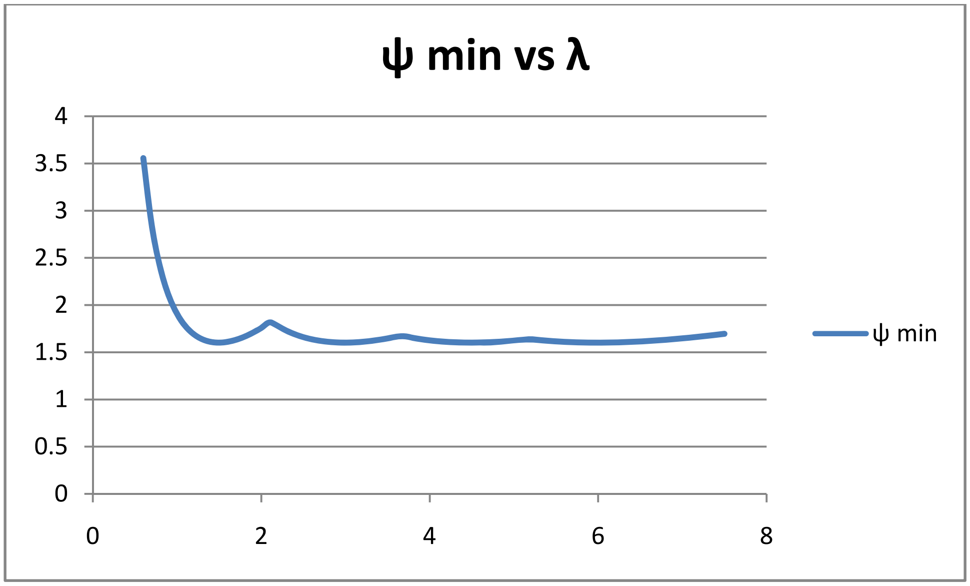


**Fig1.3:** variation of buckling load coefficient $\psi$ with apect ratio $\lambda$ for $\nu_x = 0.25$ and $c = 0.2$

## 1.3 Effective thickness of a stiffened plate

Now consider a stiffened plate with given dimensions as shown in the figure.

formula for effective thickness of this stiffened plate $h_{eff}$ is derived.

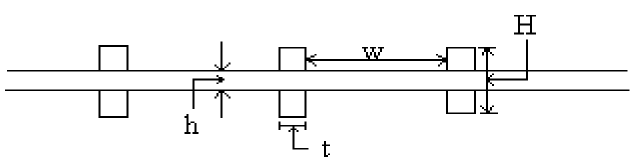


**Fig 1.4:** Stiffened Plate

Let $I_1$ be the moment of inertia of plate and $I_2$ be the moment of inertia of stiffener on two sides with portion of plate between them.

$$\left(N_x\right)_{cr} = \frac{\pi^2}{b^2}\left[2\sqrt{D_x D_y} + \left(D_x \nu_y + D_y \nu_x\right) + 2D_{xy}\right] \quad (1.25)$$

$$D_{xy} = \frac{Eh^3}{12(1+\nu)} = \frac{EI_1}{w(1+\nu)} \quad (1.26a)$$

$$\text{where } I_1 = \frac{wh^3}{12}$$

$$D_x = \frac{Eh^3}{12(1-\nu^2)} = \frac{EI_1}{w(1-\nu^2)} \tag{1.26b}$$

$$D_y = \frac{Eh^3}{12(1-\nu^2)} + \frac{EI_2}{w}$$

$$\text{where } I_2 = \frac{tH^3}{12}$$

$$D_y = \frac{EI_1}{w(1-\nu^2)} + \frac{EI_2}{w} \tag{1.26c}$$

$$D_x + D_y = 2\frac{EI_1}{w(1-\nu^2)} + \frac{EI_2}{w}$$

$$= \frac{EI_1}{w(1-\nu^2)}\left[2 + \frac{I_2}{I_1}\left(1-\nu^2\right)\right] \tag{1.27}$$

$$\sqrt{D_x D_y} = \frac{EI_1}{w(1-\nu^2)}\sqrt{1 + \frac{I_2}{I_1}\left(1-\nu^2\right)}$$

Substituting in the expression for $\left(N_x\right)_{cr}$

$$\left(N_x\right)_{cr} = \frac{\pi^2}{b^2}\left(\frac{Eh^3}{12(1-\nu^2)}\right)\left[2\sqrt{\left(1 + \frac{I_2}{I_1}(1-\nu^2)\right)} + \nu\left(2 + \frac{I_2}{I_1}(1-\nu^2)\right) + 2(1-\nu)\right]$$

Comparing with the expression for isotropic plate,

$$\left(\frac{h_{eff}}{h}\right)^3 = \frac{1}{2}\left[\sqrt{\left(1 + \frac{I_2}{I_1}(1-\nu^2)\right)} + \frac{\nu}{2}\frac{I_2}{I_1}(1-\nu^2) + 1\right] \tag{1.28}$$

## 1.4 Comparison

Closely spaced stiffened plate has been considered here and then treating that as an orthotropic plate code gives the following formula.

$$h_{eff} = \sqrt[3]{\frac{12I_s}{w}}$$

Where $Is$ = moment of inertia of the full area of the multiple stiffened element, including the intermediate stiffeners, about its own centroidal axis.

For our case $I_s = I_2 + I_1$

The above equation can be written as $(h_{eff})^3 = 12I_s/w$

Also we have $I_1 = wh^3/12$

So $(h_{eff}/h)^3 = 1 + I_2/I_1$

Also as derived earlier $$\left(\frac{h_{eff}}{h}\right)^3 = \frac{1}{2}\left[\sqrt{\left(1+\frac{I_2}{I_1}(1-\nu^2)\right)} + \frac{\nu}{2}\frac{I_2}{I_1}(1-\nu^2)+1\right]$$

This formula is compared with the one derived earlier for stiffened plate. Typical values have been taken to calculate ratio. Although two formulae can not be compared directly as there are restrictions on code formula under which it can be applied.

The curve of $h_{eff}/h$ vs $H/h$ for code formula is as shown on the next page.

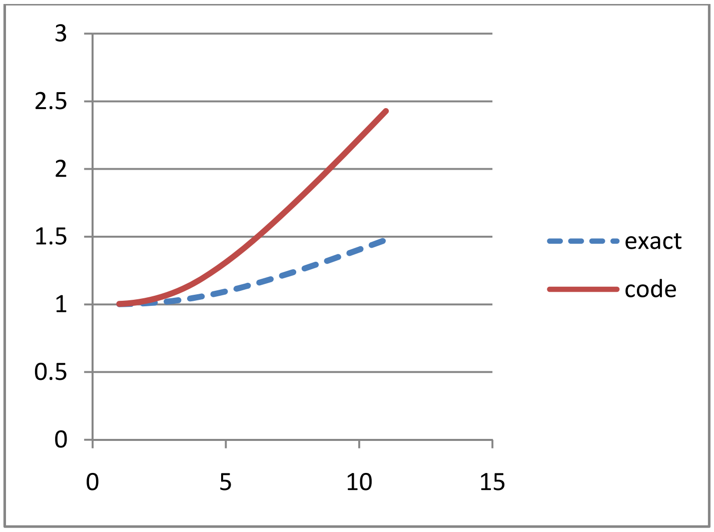


**Fig1.5:** $h_{eff}/h$ vs. $H/h$ for code formula

As we can see two values are close when H/h ($I_2$/ $I_1$) is low but diverges as we increase the ratio H/h($I_2$/ $I_1$)

## 1.5 Corrugated Plate

A stiffened orthotropic plate can be modeled as a corrugated plate. Approximate flexural rigidities have been considered in different directions and then final formula for effective width of corrugated plate has been calculated.

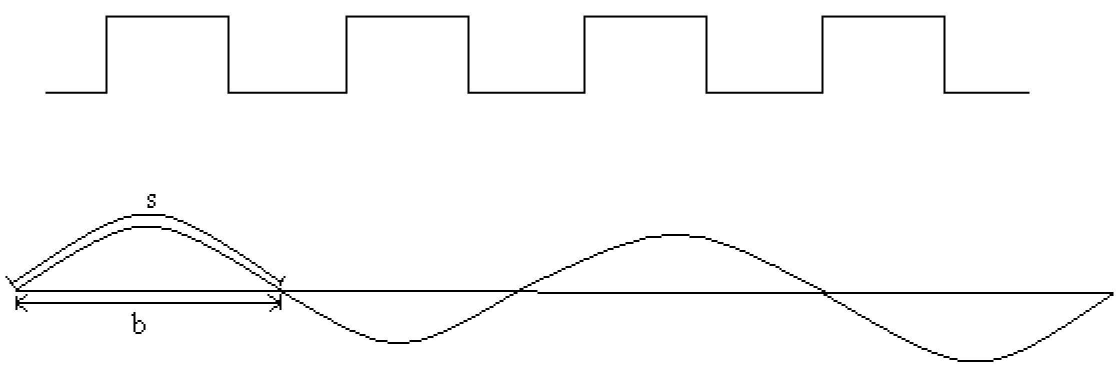


**Fig 1.6:** Corrugated Plate

Modeling as a corrugated plate and taking flexural rigidity different in X and Y direction

We have

$$D_y \approx \frac{b}{2s} D$$

$$D_x \approx \frac{\mathrm{I}s}{\mathrm{bh}^3/12} D$$

$$D_{xy} \approx \frac{\mathrm{b}}{\mathrm{s}}(1-\nu)D$$

$$\nu_x D_y \approx \frac{b}{s} \nu D$$

$$\nu_x D_y + D_{xy} \approx \frac{b}{s} D$$

Now substituting these values in the equation

$$(N_x)_{cr} = \frac{\pi^2}{b^2}\left[2\sqrt{D_x D_y} + (D_x \nu_y + D_y \nu_x) + 2D_{xy}\right]$$

And simplifying, following result is derived

$$\frac{h_{eq}}{h} \approx \left[\frac{b}{2s} + \left(\frac{3I_s}{sh^3}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}$$

# Chapter 2

# Post Buckling Strength: Stress Solution

## 2.1 Effective width concept in thin walled steel section

The theoretical critical load for plate is not necessarily is a satisfactory basis for design, since the ultimate strength can be much greater than the critical buckling load. A plate loaded in uniaxial compression will undergo stress redistribution as well as develop transverse tensile membrane stresses after buckling that provide post buckling support. Thus additional load often be applied without structural damage.

### 2.1.1 Postbuckling Strength

Postbuckling strength in plates is mainly due to the redistribution of axial compressive stresses. Local buckling causes a loss of stiffness and a redistribution of stresses. Uniform edge compression in the longitudinal direction results in nonuniform stress distribution after buckling and buckled plate derives almost all of its stiffness from the longitudinal edge supports.

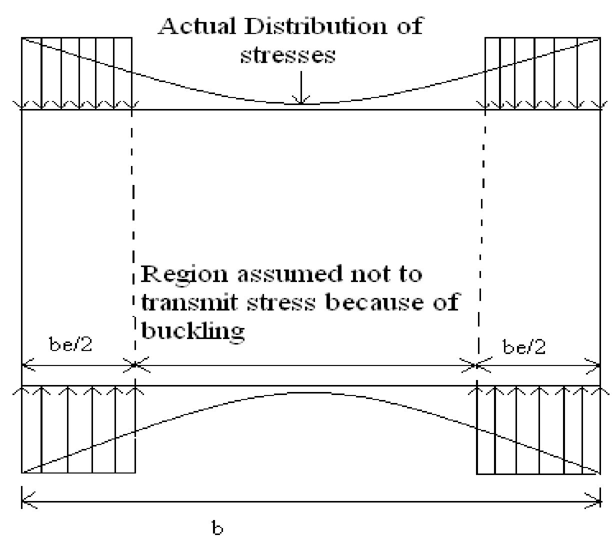

**Fig 2.1:** Postbuckling Stress Distribution

The fact that much of the load carried by the region of the plate in the close vicinity of the edges suggests the use of "*effective width concept*". The maximum strength of plates can be estimated by the use of *effective width* concept.

This concept makes a simplifying assumption that maximum edge stress acts uniformly over two "strips" of the plate and the central region remains unstressed. Thus only a fraction of the width is considered to be effective in resisting the applied compression.

## 2.2 **Stress Solution**

The equilibrium equation in the z-direction is obtained as

$$D_x \frac{\partial^4 w}{\partial x^4} + \left(\nu_x D_y + \nu_y D_x + 2D_{xy}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - \left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}\right) = 0 \tag{2.1}$$

### 2.2.1 Compatibility

The compatibility relations are obtained as

$$\varepsilon_{x0} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{2.2a}$$

$$\varepsilon_{y0} = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{2.2b}$$

and

$$\gamma_{xy0} = \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{2.2c}$$

Where $\varepsilon_{x0}$, $\varepsilon_{y0}$ and $\gamma_{xy0}$ are middle surface strains in *x* and *y* directions and in shear respectively. $u_0$ and $v_0$ are displacements at the middle surface and $w$ is the displacement in transverse direction.

Strains can be related to forces as

$$\varepsilon_{x0} = \frac{1}{E_x h}\left(N_x - \nu_x N_y\right) \tag{2.3a}$$

$$\varepsilon_{y0} = \frac{1}{E_y h}\left(N_y - \nu_y N_x\right) \tag{2.3b}$$

$$\gamma_{xy0} = \frac{1}{h}\frac{N_{xy}}{G_{xy}} \tag{2.3c}$$

Differentiating (2.2a) twice w.r.t. *y*, (2.2b) twice w.r.t. *x* and (2.2c) successively w.r.t. *x* and *y* and combining the results

$$\frac{\partial^2 \varepsilon_{x0}}{\partial y^2} + \frac{\partial^2 \varepsilon_{y0}}{\partial x^2} - \frac{\partial^2 \gamma_{xy0}}{\partial x \partial y} = \left[\frac{\partial^2 w}{\partial x \partial y}\right]^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{2.4}$$

This is deformation compatibility equation. To reduce the number of equations that must be solved, a stress function is introduced. Let the in-plane forces be defined in terms of a function $F(x, y)$ as

$$N_x = h\frac{\partial^2 F}{\partial y^2} \tag{2.5a}$$

$$N_y = h\frac{\partial^2 F}{\partial x^2} \tag{2.5b}$$

$$N_{xy} = -h\frac{\partial^2 F}{\partial x \partial y} \tag{2.5c}$$

Using equations (2.3a) to (2.3c)

$$\varepsilon_{x0} = \frac{1}{E_x}\left(\frac{\partial^2 F}{\partial y^2} - \nu_x \frac{\partial^2 F}{\partial x^2}\right) \tag{2.6a}$$

$$\varepsilon_{y0} = \frac{1}{E_y}\left(\frac{\partial^2 F}{\partial x^2} - \nu_y \frac{\partial^2 F}{\partial y^2}\right) \tag{2.6b}$$

$$\gamma_{xy0} = -\frac{1}{G_{xy}}\frac{\partial^2 F}{\partial x \partial y} \tag{2.6c}$$

substituting from (2.6a), (2.6b) and (2.6c) into (2.4) and from (2.5a), (2.5b) and (2.5c) into(2.1) gives

$$\frac{1}{E_x}\left[\frac{\partial^4 F}{\partial y^4} - \nu_x \frac{\partial^4 F}{\partial x^2 \partial y^2}\right] + \frac{1}{E_y}\left[\frac{\partial^4 F}{\partial x^4} - \nu_y \frac{\partial^4 F}{\partial x^2 \partial y^2}\right] + \frac{h}{G_{xy}}\frac{\partial^4 F}{\partial x^2 \partial y^2} = \left[\frac{\partial^2 w}{\partial x \partial y}\right]^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

or

$$\frac{1}{E_y}\frac{\partial^4 F}{\partial x^4} + \left[\frac{h}{G_{xy}} - \frac{\nu_x}{E_x} - \frac{\nu_y}{E_y}\right]\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_x}\frac{\partial^4 F}{\partial y^4} = \left[\frac{\partial^2 w}{\partial x \partial y}\right]^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{2.7a}$$

and

$$D_x \frac{\partial^4 w}{\partial x^4} + \left(\nu_x D_y + \nu_y D_x + 2D_{xy}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - \left(h\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + h\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2h\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right) = 0$$

(2.7b)

## 2.3 Post buckling behavior of axially compressed plates

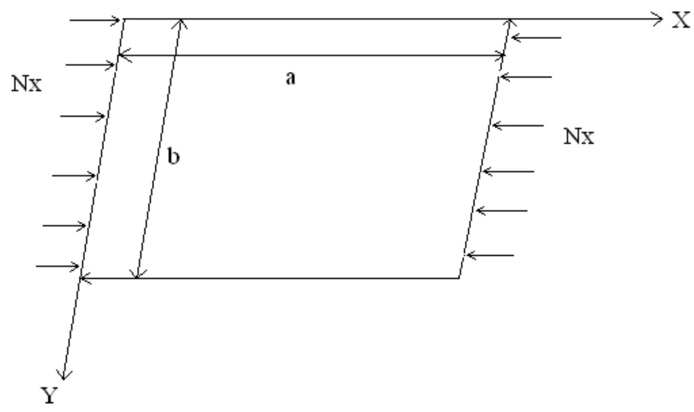


**Fig2.2:** Simply supported plate compressed in *x*-direction

### 2.3.1 Boundary conditions

Transverse boundary conditions corresponding to simply supported edges are

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0, a$$

$$w = \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = 0, b$$

Thus

Average value of applied compressive stress $$\sigma_{xa} = -\frac{1}{ah}\int_0^a N_x dy \qquad (2.8)$$

Assuming lateral deflection function as $$w = f \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (2.9)$$

and substituting in the equation (2.7a)

$$\frac{1}{E_y}\frac{\partial^4 F}{\partial x^4} + \left[\frac{h}{G_{xy}} - \frac{\nu_x}{E_x} - \frac{\nu_y}{E_y}\right]\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_x}\frac{\partial^4 F}{\partial y^4} = \left[\frac{\partial^2 w}{\partial x \partial y}\right]^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

which reduces to

$$\frac{1}{E_y}\frac{\partial^4 F}{\partial x^4}+\left[\frac{h}{G_{xy}}-\frac{\nu_x}{E_x}-\frac{\nu_y}{E_y}\right]\frac{\partial^4 F}{\partial x^2 \partial y^2}+\frac{1}{E_x}\frac{\partial^4 F}{\partial y^4}=\frac{f^2 m^2 n^2 \pi^4}{2a^2 b^2}\left[\cos\frac{2m\pi x}{a}+\cos\frac{2n\pi y}{b}\right] \quad (2.10)$$

Solution of the equation consists of a homogeneous part and a particular part i.e.

$F = F_h + F_p$

For homogeneous part of solution, RHS has to be set equal to zero. But this will be equivalent to letting $w = 0$. Therefore the homogeneous solution corresponds to the in-plane stress distribution in the plate just prior to buckling. But at that instant $N_x$ will be uniform and $N_y$ and $N_{xy}$ will be zero. Therefore a homogeneous solution of the equation will be

$$F_h = Ay^2 \quad (2.11)$$

as $N_x = -\sigma_{xa} h$

we get

$$F_h = -\frac{\sigma_{xa}}{2} y^2 \quad (2.12)$$

Now corresponding to the RHS, the particular solution can be written as

$$F_p = B\cos\frac{2m\pi x}{a} + C\cos\frac{2n\pi y}{b} \quad (2.13)$$

Substituting back in the differential equation and comparing the coefficients of the terms $\cos\frac{2m\pi x}{a}$ and $\cos\frac{2n\pi y}{b}$ ,values of B and C are obtained as

$$B = \frac{E_y f^2 a^2 n^2}{32 m^2 b^2} \quad \text{and} \quad C = \frac{E_x f^2 b^2 m^2}{32 n^2 a^2} \quad (2.14)$$

using which complete solution of $F$ is obtained as

$$F = \frac{E_y f^2 a^2 n^2}{32 m^2 b^2}\cos\frac{2m\pi x}{a} + \frac{E_x f^2 b^2 m^2}{32 n^2 a^2}\cos\frac{2n\pi y}{b} - \frac{\sigma_{xa} y^2}{2} \quad (2.15)$$

## 2.4 Using Galerkin method for finding f

$$w = f \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

The Galerkin equation is

$$\int_0^a\int_0^a Q(f)g(x,y)dxdy = 0 \tag{2.16}$$

Where on arranging the terms *Q(f)* is obtained as

$$Q(f)=\begin{bmatrix}\pi^4 f\left(\dfrac{D_x m^4}{a^4}+\dfrac{D_y n^4}{b^4}+\left(\nu_x D_y+\nu_y D_x+2D_{xy}\right)\dfrac{m^2n^2}{a^2b^2}\right)+\\ \left(\dfrac{E_y hf^3 n^4\pi^4}{8b^4}Cos\dfrac{2\pi mx}{a}+\dfrac{E_x hf^3 m^4\pi^4}{8a^4}Cos\dfrac{2\pi ny}{b}\right)+\sigma_{xa}hf\dfrac{\pi^2m^2}{a^2}\end{bmatrix}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{2.17}$$

Substituting the value of $Q(f)$ in Galerkin equation, taking $g(x,y)= f\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$ and integrating with proper limits the following is obtained

$$\sigma_{xa}=\frac{\pi^2m^2a^2}{h}\left[\frac{D_x m^4}{a^4}+\frac{D_y n^4}{b^4}+\left(\nu_x D_y+\nu_y D_x+2D_{xy}\right)\frac{m^2n^2}{a^2b^2}\right]+\frac{\pi^2f^2a^2}{16}\left[\frac{E_x m^4}{a^4}+\frac{E_y n^4}{b^4}\right]$$

which can also be expressed as

$$\sigma_{xa}=\sigma_{cr}+\frac{\pi^2f^2a^2}{16}\left[\frac{E_x m^4}{a^4}+\frac{E_y n^4}{b^4}\right] \tag{2.18}$$

where

$$\sigma_{cr}=\frac{\pi^2m^2a^2}{h}\left[\frac{D_x m^4}{a^4}+\frac{D_y n^4}{b^4}+\left(\nu_x D_y+\nu_y D_x+2D_{xy}\right)\frac{m^2n^2}{a^2b^2}\right] \tag{2.19}$$

Now as

$$\sigma_{xx} = -\frac{\partial^2 F}{\partial y^2}$$

substituting the value of $F$ , $\sigma_{xx}$ is obtained as

$$\sigma_{xx} = \frac{E_x \pi^2 m^2 f^2}{8a^2} Cos \frac{2\pi ny}{b} + \sigma_{xa} \tag{2.20}$$

Also we have
$$\frac{\pi^2 f^2}{8} = \frac{2(\sigma_{xx} - \sigma_{cr})}{\frac{a^2}{m^2}\left[\frac{E_x m^4}{a^4} + \frac{E_y n^4}{b^4}\right]}$$

so substituting the value in the equation (2.20), $\sigma_{xx}$ is obtained as

$$\sigma_{xx} = \sigma_{xa} + \frac{2(\sigma_{xx} - \sigma_{cr})}{\left[1 + \frac{a^4}{m^4}\frac{n^4}{b^4}\frac{E_y}{E_x}\right]} Cos \frac{2\pi ny}{b}$$

Now let $\frac{m\pi}{a} = M$ and $\frac{n\pi}{b} = N$

Then the equation becomes

$$\sigma_{xx} = \sigma_{xa} + \frac{2(\sigma_{xx} - \sigma_{cr})}{\left[1 + \frac{N^4}{M^4}\frac{E_y}{E_x}\right]} Cos \frac{2\pi ny}{b} \tag{2.21}$$

## 2.5 Finding Effective width: Stress Method

$$b_e \sigma_{\max} = \int_0^b \sigma_{xx} dy \tag{2.22}$$

$$\sigma_{\max} = \sigma_{xx}\big|_{y=0} = \sigma_{xa} + \frac{2(\sigma_{xx} - \sigma_{cr})}{\left[1 + \frac{a^4}{m^4}\frac{n^4}{b^4}\frac{E_y}{E_x}\right]}$$

now,

$$\int_0^b \sigma_{xx} dy = \sigma_{xa} b + (........)\int_0^b Cos\frac{2\pi ny}{b}dy$$

so

$$\int_0^b \sigma_{xx} \partial y = \sigma_{xa} b$$

hence

$$\frac{b_e}{b} = \frac{\sigma_{xa}}{\sigma_{xa} + \frac{2(\sigma_{xa} - \sigma_{cr})}{(1+\eta)}} \tag{2.23}$$

where

$$\eta = \frac{a^4}{m^4}\frac{n^4}{b^4}\frac{E_y}{E_x} \tag{2.24}$$

Now we have

$$\frac{\sigma_{xa}}{\sigma_{xx}} = \frac{P}{P_{cr}}$$

so

$$\frac{b_e}{b} = \frac{\frac{P}{P_{cr}}}{\frac{P}{P_{cr}} + \frac{2(\frac{P}{P_{cr}} - 1)}{(1+\eta)}} \tag{2.25}$$

# Chapter 3

# Post Buckling Strength: Method of Successive Approximations

## 3.1 Introduction

In the method of successive approximations, the set of Von Karman large deflection equations for the plates are converted from three nonlinear partial differential equations into infinite number of linear partial differential equations by expanding the displacement terms into a power series form of a parameter. The first few equations of the infinite set of equations are obtained as small deflection equations which give the solution for pre-buckling stage. Further solution of more equations gives approximate solution for post-buckling range.

The study of post-buckling behaviour of a simply supported orthotropic plate subjected to longitudinal compression is presented here.

## 3.2 Problem background

For a plate with no lateral loads, Von Karman equations for large deflection can be written as

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \qquad (3.1a)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \qquad (3.1b)$$

$$D_x \frac{\partial^4 w}{\partial x^4} + \left(\nu_x D_y + \nu_y D_x + 2D_{xy}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0 \qquad (3.1c)$$

The strain force relations are the following

$$\varepsilon_x = \frac{1}{h}\left(\frac{N_x}{E_x} - \nu_y \frac{N_y}{E_y}\right) \qquad (3.2a)$$

$$\varepsilon_y = \frac{1}{h}\left(\frac{N_y}{E_y} - \nu_x \frac{N_x}{E_x}\right) \qquad (3.2b)$$

$$\gamma_{xy} = \frac{1}{h}\frac{N_{xy}}{G_{xy}} \tag{3.2c}$$

From equation (2a), (2b) & (2c), forces can be expressed in terms of strains as

$$N_x = \frac{E_x h}{1-\nu_x \nu_y}(\varepsilon_x + \nu_y \varepsilon_y) \tag{3.3a}$$

$$N_y = \frac{E_y h}{1-\nu_x \nu_y}(\varepsilon_y + \nu_x \varepsilon_x) \tag{3.3b}$$

$$N_{xy} = G_{xy} h \gamma_{xy} \tag{3.3c}$$

Strain-displacement equations are

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{3.4a}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{3.4b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{3.4c}$$

substituting these relations we get the force-displacement relations as

$$N_x = \frac{E_x h}{1-\nu_x \nu_y}\left(\frac{\partial u}{\partial x} + \nu_y \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \nu_y \left(\frac{\partial w}{\partial y}\right)^2\right\}\right) \tag{3.5a}$$

$$N_y = \frac{E_y h}{1-\nu_x \nu_y}\left(\frac{\partial v}{\partial y} + \nu_x \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial w}{\partial y}\right)^2 + \nu_x \left(\frac{\partial w}{\partial x}\right)^2\right\}\right) \tag{3.5b}$$

$$N_{xy} = G_{xy} h\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \tag{3.5c}$$

Now assuming that $u$, $v$ & $w$ may be expanded in a power series in terms of an arbitrary parameter $\alpha$, they can be expressed as

$$u = \sum_{n=0,2}^{\infty} u_n \alpha^n \tag{3.6a}$$

$$v = \sum_{n=0,2}^{\infty} v_n \alpha^n \tag{3.6b}$$

$$w = \sum_{n=1,3}^{\infty} w_n \alpha^n \tag{3.6c}$$

Here $u$ and $v$ are assumed to start from a term having zero power of $\alpha$ whereas $w$ is assumed to start with non-zero power of $\alpha$. The reason is that just prior to buckling, the in-plane deflections $u$ and $v$ are expected to have finite value whereas the transverse deflection $w$ is expected to be zero until the plate buckles.

The plate can buckle in either direction but $w(x, y)$ should be independent of that except for a sign. Hence for + ve to –ve values of $\alpha$, $w$ should just only change its sign and therefore only odd powers of $\alpha$ are assumed in the power series expansion of $w$.

Opposite to this, the in-plane deflections $u$ and $v$ should be independent of the direction of buckling (and hence of the sign of $\alpha$). So only even powers of $\alpha$ are assumed in the power series expansion of $u$ and $v$.

Substituting these in equations (5a) to (5c)

$$N_x = \sum_{n=0,2}^{\infty}\left(\frac{E_x h}{1-\nu_x \nu_y}\right)\left(\frac{\partial u_n}{\partial x} + \nu_y \frac{\partial v_n}{\partial y}\right)\alpha^n + \sum_{m=1,3}^{\infty}\sum_{n=1,3}^{\infty}\frac{1}{2}\left(\frac{E_x h}{1-\nu_x \nu_y}\right)\left(\frac{\partial w_m}{\partial x}\frac{\partial w_n}{\partial x} + \nu_y \frac{\partial w_m}{\partial y}\frac{\partial w_n}{\partial y}\right)\alpha^{m+n}$$

$$N_y = \sum_{n=0,2}^{\infty}\left(\frac{E_y h}{1-\nu_x \nu_y}\right)\left(\frac{\partial v_n}{\partial y} + \nu_x \frac{\partial u_n}{\partial x}\right)\alpha^n + \sum_{m=1,3}^{\infty}\sum_{n=1,3}^{\infty}\frac{1}{2}\left(\frac{E_y h}{1-\nu_x \nu_y}\right)\left(\frac{\partial w_m}{\partial y}\frac{\partial w_n}{\partial y} + \nu_x \frac{\partial w_m}{\partial x}\frac{\partial w_n}{\partial x}\right)\alpha^{m+n}$$

$$N_{xy} = \sum_{n=0,2}^{\infty} G_{xy} h\left(\frac{\partial u_n}{\partial y} + \frac{\partial v_n}{\partial x}\right)\alpha^n + \sum_{m=1,3}^{\infty}\sum_{n=1,3}^{\infty} G_{xy} h\left(\frac{\partial w_m}{\partial x}\frac{\partial w_n}{\partial y}\right)\alpha^{m+n}$$

Expressing these equations as sum of two terms as

$$N_x = \sum_{n=0,2}^{\infty} N_x^{(n)} \alpha^n + \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} N_x^{(m,n)} \alpha^{m+n} \tag{3.7a}$$

$$N_y = \sum_{n=0,2}^{\infty} N_y^{(n)} \alpha^n + \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} N_y^{(m,n)} \alpha^{m+n} \tag{3.7b}$$

$$N_{xy} = \sum_{n=0,2}^{\infty} N_{xy}^{(n)} \alpha^n + \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} N_{xy}^{(m,n)} \alpha^{m+n} \tag{3.7c}$$

where

$$N_x^{(n)} = \left(\frac{E_x h}{1-\nu_x \nu_y}\right)\left(\frac{\partial u_n}{\partial x} + \nu_y \frac{\partial v_n}{\partial y}\right) \tag{3.8a}$$

$$N_x^{(m,n)} = \frac{1}{2}\left(\frac{E_x h}{1-\nu_x \nu_y}\right)\left(\frac{\partial w_m}{\partial x}\frac{\partial w_n}{\partial x} + \nu_y \frac{\partial w_m}{\partial y}\frac{\partial w_n}{\partial y}\right) \tag{3.8b}$$

$$N_y^{(n)} = \left(\frac{E_y h}{1-\nu_x \nu_y}\right)\left(\frac{\partial v_n}{\partial y} + \nu_x \frac{\partial u_n}{\partial x}\right) \tag{3.8c}$$

$$N_y^{(m,n)} = \frac{1}{2}\left(\frac{E_y h}{1-\nu_x \nu_y}\right)\left(\frac{\partial w_m}{\partial y}\frac{\partial w_n}{\partial y} + \nu_x \frac{\partial w_m}{\partial x}\frac{\partial w_n}{\partial x}\right) \tag{3.8d}$$

$$N_{xy}^{(n)} = G_{xy} h\left(\frac{\partial u_n}{\partial y} + \frac{\partial v_n}{\partial x}\right) \tag{3.8e}$$

$$N_{xy}^{(m,n)} = G_{xy} h\left(\frac{\partial w_m}{\partial x}\frac{\partial w_n}{\partial y}\right) \tag{3.8f}$$

Now as stated earlier in this chapter, for a plate with no lateral loads, Von Karman large deflection equations can be written as

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$D_x \frac{\partial^4 w}{\partial x^4} + \left(\nu_x D_y + \nu_y D_x + 2D_{xy}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0$$

Substituting the power series expansions of all the terms in these equations, three equations are obtained each of which equates a power series of $\alpha$ to zero. For the equation to hold true for all values of $\alpha$, coefficients of all the exponents of $\alpha$ in each of the three equations should be equal to zero.

Equating the coefficients of $\alpha^0 = 0$ ; the following relations are obtained

$$\left.\begin{aligned} \frac{\partial N_x^{(0)}}{\partial x} + \frac{\partial N_{xy}^{(0)}}{\partial y} = 0 \\ \frac{\partial N_{xy}^{(0)}}{\partial x} + \frac{\partial N_y^{(0)}}{\partial y} = 0 \end{aligned}\right\} \tag{3.9a}$$

and equating the coefficients of $\alpha^1 = 0$ ;

$$D_x \frac{\partial^4 w_1}{\partial x^4} + \left(\nu_x D_y + \nu_y D_x + 2D_{xy}\right)\frac{\partial^4 w_1}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w_1}{\partial y^4} - \left(N_x^{(0)} \frac{\partial^2 w_1}{\partial x^2} + N_y^{(0)} \frac{\partial^2 w_1}{\partial y^2} + 2N_{xy}^{(0)} \frac{\partial^2 w_1}{\partial x \partial y}\right) = 0 \tag{3.9b}$$

Similarly equating the co-efficient of $\alpha^2 = 0$

$$\left.\begin{aligned} \frac{\partial N_x^{(2)}}{\partial x} + \frac{\partial N_x^{(1,1)}}{\partial x} + \frac{\partial N_{xy}^{(2)}}{\partial y} + \frac{\partial N_{xy}^{(1,1)}}{\partial y} = 0 \\ \frac{\partial N_y^{(2)}}{\partial y} + \frac{\partial N_y^{(1,1)}}{\partial y} + \frac{\partial N_{xy}^{(2)}}{\partial x} + \frac{\partial N_{xy}^{(1,1)}}{\partial x} = 0 \end{aligned}\right\} \tag{3.9c}$$

and equating the co-efficient of $\alpha^3 = 0$ the next equation is obtained as

$$D_x \frac{\partial^4 w_3}{\partial x^4} + \left(\nu_x D_y + \nu_y D_x + 2D_{xy}\right)\frac{\partial^4 w_3}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w_3}{\partial y^4} - \left(N_x^{(0)} \frac{\partial^2 w_3}{\partial x^2} + N_y^{(0)} \frac{\partial^2 w_3}{\partial y^2} + N_{xy}^{(0)} \frac{\partial^2 w_3}{\partial x \partial y}\right) = \left[N_x^{(2)} + N_x^{(1,1)}\right]\frac{\partial^2 w_1}{\partial x^2} + \left[N_y^{(2)} + N_y^{(1,1)}\right]\frac{\partial^2 w_1}{\partial y^2} + \left[N_{xy}^{(2)} + N_{xy}^{(1,1)}\right]\frac{\partial^2 w_1}{\partial x \partial y} \tag{3.9d}$$

Similarly equating higher coefficients of $\alpha$ to zero, more equations can be obtained.

## 3.3 Compressive load problem

The plate under compressive loading along x- direction has the length $a$, width $b$ and thickness $h$. The origin lies at one corner of the plate as shown in figure. The edges are all simply supported with compressive load of $N_x$ per unit width.

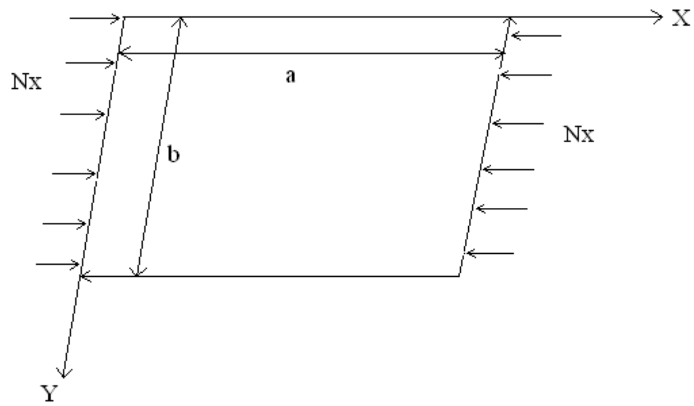


**Fig: 3.1-** Loading condition on the plate

### 3.3.1 Boundary Conditions

Zero transverse deflection at the edges

$$w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0$$

Zero moment at edges

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{(0,y)} = \left(\frac{\partial^2 w}{\partial x^2}\right)_{(a,y)} = \left(\frac{\partial^2 w}{\partial x^2}\right)_{(x,0)} = \left(\frac{\partial^2 w}{\partial x^2}\right)_{(x,b)} = 0$$

Constant displacement along an edge

$$\left(\frac{\partial u}{\partial y}\right)_{(0,y)} = \left(\frac{\partial u}{\partial y}\right)_{(a,y)} = \left(\frac{\partial v}{\partial x}\right)_{(x,0)} = \left(\frac{\partial v}{\partial x}\right)_{(x,b)} = 0$$

Zero shear stress at edges

$$\left(\frac{\partial u}{\partial y}\right)_{(x,0)} = \left(\frac{\partial u}{\partial y}\right)_{(x,b)} = \left(\frac{\partial v}{\partial x}\right)_{(0,y)} = \left(\frac{\partial v}{\partial x}\right)_{(a,y)} = 0$$

Loaded edges

$$\int_0^b (N_x)dy = -P \qquad \text{at } x = 0, a$$

Unloaded edges

$$\int_0^a (N_y)dx = -P \qquad \text{at } y = 0, b$$

The load is equal to or greater than the buckling load. Substituting the relations from equation (8a) & (8c) into the condition for loaded edge, the following is obtained

$$P = \sum_{n=0,2}^{\infty} P^{(n)} \alpha^n \tag{3.10}$$

where

For loaded edges

$$P^{(0)} = -\int_0^b \left(N_x^{(0)}\right)dy \qquad \text{at } x = 0, a \tag{3.11a}$$

$$P^{(2)} = -\int_0^b \left(N_x^{(2)} + N_x^{(1,1)}\right)dy \qquad \text{at } x = 0, a \tag{3.11b}$$

$$P^{(4)} = -\int_0^b \left(N_x^{(4)} + 2N_x^{(1,3)}\right)dy \qquad \text{at } x = 0, a \tag{3.11c}$$

(as $N_x^{(1,3)} = N_x^{(3,1)}$)

Similarly for unloaded edges

$$\int_0^a \left(N_y^{(0)}\right)dx = 0 \qquad \text{at } y = 0, b \qquad (3.12a)$$

$$\int_0^a \left(N_y^{(2)} + N_y^{(1,1)}\right)dx = 0 \qquad \text{at } y = 0, b \qquad (3.12b)$$

$$\int_0^a \left(N_y^{(4)} + 2N_y^{(1,3)}\right)dx = 0 \qquad \text{at } y = 0, b \qquad (3.12c)$$

Now using equations (1a) and (1b)

$$\frac{\partial N_x^{(0)}}{\partial x} + \frac{\partial N_{xy}^{(0)}}{\partial y} = 0$$

and

$$\frac{\partial N_{xy}^{(0)}}{\partial x} + \frac{\partial N_y^{(0)}}{\partial y} = 0$$

Substituting the relations from equations (3.8a), (3.8c) and (3.8e)

$$\frac{\partial}{\partial x}\left[\left(\frac{E_x h}{1-\nu_x\nu_y}\right)\left(\frac{\partial u_0}{\partial x} + \nu_y \frac{\partial v_0}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[G_{xy} h\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)\right] = 0$$

and

$$\frac{\partial}{\partial x}\left[G_{xy} h\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\left(\frac{E_y h}{1-\nu_x\nu_y}\right)\left(\frac{\partial v_0}{\partial y} + \nu_x \frac{\partial u_0}{\partial x}\right)\right] = 0$$

rewriting these

$$\left(\frac{E_x}{1-\nu_x\nu_y}\right)\frac{\partial^2 u_0}{\partial x^2} + G_{xy}\frac{\partial^2 u_0}{\partial y^2} + \left(\frac{E_x\nu_y}{1-\nu_x\nu_y} + G_{xy}\right)\frac{\partial^2 v_0}{\partial x \partial y} = 0 \qquad (3.13a)$$

$$G_{xy}\frac{\partial^2 v_0}{\partial x^2} + \left(\frac{E_y}{1-\nu_x\nu_y}\right)\frac{\partial^2 v_0}{\partial y^2} + \left(\frac{E_y\nu_x}{1-\nu_x\nu_y} + G_{xy}\right)\frac{\partial^2 u_0}{\partial x \partial y} = 0 \qquad (3.13b)$$

Solutions of these equations satisfying the boundary conditions will be of the form

$$u_0 = U_0\left(x - \frac{a}{2}\right)$$

$$v_0 = V_0\left(y - \frac{b}{2}\right)$$

Using the loading boundary conditions the values of $U_0$ and $V_0$ are obtained as

$$U_0 = -\frac{P^{(0)}}{E_x hb}$$

and $$V_0 = \frac{\nu_x P^{(0)}}{E_x hb}$$

so that

$$u_0 = -\frac{P^{(0)}}{E_x hb}\left(x - \frac{a}{2}\right) \tag{3.14a}$$

$$v_0 = \frac{\nu_x P^{(0)}}{E_x hb}\left(y - \frac{b}{2}\right) \tag{3.14b}$$

and therefore

$$\begin{aligned} N_x^{(0)} &= -\frac{P^{(0)}}{b} \\ N_y^{(0)} &= N_{xy}^{(0)} = 0 \end{aligned} \tag{3.15}$$

Now $w_1$ can be obtained from the equation (3.9b) which has the solution of the form

$$w_1 = W_1 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{3.16}$$

$$= W_1 \sin Mx \sin Ny$$

which satisfies the boundary conditions. Putting this solution in the equation (3.9b) and substituting the results of the equation (3.15)

$$D_x \frac{m^4\pi^4}{a^4} W_1 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + \left(\nu_x D_y + \nu_y D_x + 2D_{xy}\right)\frac{m^2 n^2 \pi^4}{a^2 b^2} W_1 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} +$$
$$D_y \frac{n^4\pi^4}{b^4} W_1 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} - \left(\frac{P^{(0)}}{b}\frac{m^2\pi^2}{a^2} W_1 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right) = 0$$

which gives the value of $P^{(0)}$ as

$$P^{(0)} = \frac{b\left[D_x \frac{m^4\pi^4}{a^4} + \left(\nu_x D_y + \nu_y D_x + 2D_{xy}\right)\frac{m^2 n^2 \pi^4}{a^2 b^2} + D_y \frac{n^4\pi^4}{b^4}\right]}{\frac{m^2\pi^2}{a^2}} \tag{3.17}$$

Now using equation (3.9c)

$$\frac{\partial}{\partial x}\left[\frac{E_x h}{(1-\nu_x\nu_y)}\left(\frac{\partial u_2}{\partial x} + \nu_y \frac{\partial v_2}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\frac{E_x h}{2(1-\nu_x\nu_y)}\left[\left(\frac{\partial w_1}{\partial x}\right)^2 + \nu_y\left(\frac{\partial w_1}{\partial y}\right)^2\right]\right] +$$
$$\frac{\partial}{\partial y}\left[G_{xy} h\left(\frac{\partial u_2}{\partial x} + \nu_y \frac{\partial v_2}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[G_{xy} h \frac{\partial w_1}{\partial x}\frac{\partial w_1}{\partial y}\right] = 0$$

or

$$\left(\frac{E_x h}{(1-\nu_x\nu_y)}\right)\frac{\partial^2 u_2}{\partial x^2} + G_{xy}\frac{\partial^2 u_2}{\partial y^2} + \left(\frac{E_x \nu_y}{(1-\nu_x\nu_y)} + G_{xy}\right)\frac{\partial^2 v_2}{\partial x \partial y} = -\frac{E_x}{(1-\nu_x\nu_y)}\left[\left(\frac{\partial w_1}{\partial x}\right)\left(\frac{\partial^2 w_1}{\partial x^2}\right) + \nu_y\left(\frac{\partial w_1}{\partial y}\right)\left(\frac{\partial^2 w_1}{\partial x \partial y}\right)\right]$$
$$-G_{xy}\left[\left(\frac{\partial w_1}{\partial y}\right)\left(\frac{\partial^2 w_1}{\partial x \partial y}\right) + \left(\frac{\partial w_1}{\partial x}\right)\left(\frac{\partial^2 w_1}{\partial x^2}\right)\right]$$

Substituting the value of $w_1$ and simplifying the equation becomes

$$\left(\frac{E_x h}{(1-\nu_x\nu_y)}\right)\frac{\partial^2 u_2}{\partial x^2} + G_{xy}\frac{\partial^2 u_2}{\partial y^2} + \left(\frac{E_x \nu_y}{(1-\nu_x\nu_y)} + G_{xy}\right)\frac{\partial^2 v_2}{\partial x \partial y} =$$
$$\frac{w_1^2}{2} Sin2Mx\left[\frac{M(M^2 - \nu_y N^2)}{2}\left(\frac{E_x}{(1-\nu_x\nu_y)}\right) - M\left[\frac{(M^2 + \nu_y N^2)}{2}\left(\frac{E_x}{(1-\nu_x\nu_y)}\right) + G_{xy}N^2\right] Cos2Ny\right]$$

(3.18a)

Similarly after simplifying, the second equation of (3.9c) becomes

$$G_{xy}\frac{\partial^2 v_2}{\partial x^2}+\left(\frac{E_x h}{(1-\nu_x\nu_y)}\right)\frac{\partial^2 v_2}{\partial y^2}+G_{xy}\frac{\partial^2 u_2}{\partial y^2}+\left(\frac{E_y\nu_x}{(1-\nu_x\nu_y)}+G_{xy}\right)\frac{\partial^2 u_2}{\partial x\partial y}=$$

$$\frac{w_1^2}{2}Sin2Ny\left[\frac{N(N^2-\nu_x M^2)}{2}\left(\frac{E_y}{(1-\nu_x\nu_y)}\right)-N\left[\frac{(N^2+\nu_x M^2)}{2}\left(\frac{E_x\nu_y}{(1-\nu_x\nu_y)}\right)+G_{xy}M^2\right]\cos 2Mx\right]$$

(3.18b)

Now the total solution of the equation is given by

$$u_2=u_{2p}+u_{2h}$$

$$v_2=v_{2p}+v_{2h}$$

Total solution = Particular solution + homogeneous solution

### 3.3.2 Homogeneous solution

As the homogeneous part of the equation (3.18a) and (3.18b) in $u_2$ and $v_2$ is same as that in $u_o$ and $v_o$ in equation (3.13a) and (3.13b), the homogeneous solution will of the same form.

Let $u_2=U_2\left(x-\frac{a}{2}\right)$

& $v_2=V_2\left(x-\frac{a}{2}\right)$

Now

$$N_x^{(2)}=\frac{E_x h}{(1-\nu_x\nu_y)}\left[\frac{\partial u_2}{\partial x}+\nu_y\frac{\partial v_2}{\partial y}\right]$$

$$=\frac{E_x h}{(1-\nu_x\nu_y)}\left[U_2+\nu_y V_2\right]$$

$$N_x^{(1,1)}=\frac{E_x h}{2(1-\nu_x\nu_y)}\left[\left(\frac{\partial w_1}{\partial x}\right)^2+\nu_y\left(\frac{\partial w_1}{\partial y}\right)^2\right]$$

Substituting the value of $w_1 = W_1 SinMxSinNy$

$$N_x^{(1,1)} = \frac{E_x h}{2(1-\nu_x \nu_y)}\left[(Mw_1 CosMxSinNy)^2 + \nu_y (Nw_1 SinMxCosNy)^2\right]$$

similarly

$$N_y^{(2)} = \frac{E_y h}{(1-\nu_x \nu_y)}\left[\frac{\partial v_2}{\partial x} + \nu_x \frac{\partial u_2}{\partial y}\right]$$

$$= \frac{E_y h}{(1-\nu_x \nu_y)}\left[V_2 + \nu_y U_2\right]$$

$$N_y^{(1,1)} = \frac{E_y h}{2(1-\nu_x \nu_y)}\left[(Nw_1 SinMxCosNy)^2 + \nu_x (Mw_1 CosMxSinNy)^2\right]$$

Using the relation from equation (3.11b)

$$P^{(2)} = -\int_0^b \left(N_x^{(2)} + N_x^{(1,1)}\right)_{x=0,a} dy \qquad \text{for loaded edges}$$

$$N_x^{(2)} + N_x^{(1,1)}\Big|_{x=0,a} = \frac{E_x h}{(1-\nu_x \nu_y)}\left[U_2 + \nu_y V_2\right] + \frac{E_x h}{2(1-\nu_x \nu_y)} M^2 W_1^2 Sin^2 Ny$$

Similarly from equation (3.12b)

$$-\int_0^a \left(N_y^{(2)} + N_y^{(1,1)}\right)_{y=0,b} dx = 0 \qquad \text{for unloaded edges}$$

$$N_y^{(2)} + N_y^{(1,1)}\Big|_{y=0,b} = \frac{E_y h}{(1-\nu_x \nu_y)}\left[V_2 + \nu_x U_2\right] + \frac{E_y h}{2(1-\nu_x \nu_y)} N^2 W_1^2 Sin^2 Mx$$

Using the integrals $\int_0^b Sin^2 Nydy = \frac{b}{2}$

& $\int_0^a Sin^2 Mxdx = \frac{a}{2}$

The following is obtained

For loaded edge

$$-P^{(2)} = \frac{E_x h}{(1-\nu_x \nu_y)}\left[U_2 + \nu_y V_2\right]b + \frac{E_x h}{4(1-\nu_x \nu_y)} M^2 W_1^2 b$$

For unloaded edge

$$0 = \frac{E_y h}{(1-\nu_x \nu_y)}\left[V_2 + \nu_x U_2\right]a + \frac{E_y h}{4(1-\nu_x \nu_y)} N^2 W_1^2 a$$

Solving these two equations

$$U_2 = -\left[\frac{P^{(2)}}{E_x hb} + \frac{\left(M^2 - \nu_y N^2\right)}{\left(1-\nu_x \nu_y\right)}\frac{W_1^2}{4}\right]$$

$$V_2 = \left[\frac{\nu_x P^{(2)}}{E_x hb} + \frac{\left(\nu_x M^2 - N^2\right)}{\left(1-\nu_x \nu_y\right)}\frac{W_1^2}{4}\right]$$

### 3.3.3 Particular Solution

The particular solution will be of the form

$$u_{2p} = Sin2Mx\left[A + BCos2Ny\right]$$

$$v_{2p} = Sin2Ny\left[C + DCos2Mx\right]$$

Substituting in two equations and comparing the co-efficient of terms

$$A = -\frac{W_1^2}{16}\frac{\left(M^2 - \nu_y N^2\right)}{M} \tag{3.19a}$$

(Comparing the co-efficient of $Sin2Mx$)

$$C = -\frac{W_1^2}{16}\frac{\left(N^2 - \nu_x M^2\right)}{N} \tag{3.19b}$$

(Comparing the co-efficient of $Sin2Ny$ )

Again comparing coefficient of $Sin2MxCos2Ny$

$$B\left[\frac{M^2E_x}{(1-\nu_x\nu_y)}+N^2G_{xy}\right]+D\left[MN\left[\left(\frac{E_x\nu_y}{(1-\nu_x\nu_y)}\right)+G_{xy}\right]\right]=\frac{W_1^2}{8}\left[\frac{M(M^2+\nu_yN^2)}{(1-\nu_x\nu_y)}\frac{E_x}{(1-\nu_x\nu_y)}+G_{xy}MN^2\right]$$

comparing coefficient of $Sin2NyCos2Mx$

$$B\left[MN\left[\left(\frac{E_y\nu_x}{(1-\nu_x\nu_y)}\right)+G_{xy}\right]\right]+D\left[\frac{N^2E_y}{(1-\nu_x\nu_y)}+M^2G_{xy}\right]=\frac{W_1^2}{8}\left[\frac{N(N^2+\nu_xM^2)}{(1-\nu_x\nu_y)}\frac{E_y}{(1-\nu_x\nu_y)}+G_{xy}M^2N\right]$$

Solving above two equations $B$ and $C$ are obtained as

$$B=\frac{MW_1^2}{16} \tag{3.19c}$$

$$D=\frac{NW_1^2}{16} \tag{3.19d}$$

Therefore the complete expressions for $u_2$ and $v_2$ can be written as

$$u_2=-\left[\frac{P^{(2)}}{E_xhb}+\frac{(M^2-\nu_yN^2)}{(1-\nu_x\nu_y)}\frac{W_1^2}{4}\right]\left(x-\frac{a}{2}\right)+Sin2Mx\left[-\frac{W_1^2}{16}\frac{(M^2-\nu_yN^2)}{M}+\frac{MW_1^2}{16}Cos2Ny\right] \tag{3.20a}$$

and

$$v_2=\left[\frac{\nu_xP^{(2)}}{E_xhb}+\frac{(\nu_xM^2-N^2)}{(1-\nu_x\nu_y)}\frac{W_1^2}{4}\right]\left(x-\frac{a}{2}\right)+Sin2Ny\left[-\frac{W_1^2}{16}\frac{(N^2-\nu_xM^2)}{N}+\frac{NW_1^2}{16}Cos2Mx\right] \tag{3.20b}$$

where

$$M=\frac{m\pi}{a} \text{ and } N=\frac{n\pi}{b}$$

Now from equation (3.9d)

$$D_x \frac{\partial^4 w_3}{\partial x^4} + \left(\nu_x D_y + \nu_y D_x + 2D_{xy}\right)\frac{\partial^4 w_3}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w_3}{\partial y^4} - \left( N_x^{(0)} \frac{\partial^2 w_3}{\partial x^2} + N_y^{(0)} \frac{\partial^2 w_3}{\partial y^2} + N_{xy}^{(0)} \frac{\partial^2 w_3}{\partial x \partial y} \right) =$$
$$\left[N_x^{(2)} + N_x^{(1,1)}\right]\frac{\partial^2 w_1}{\partial x^2} + \left[N_y^{(2)} + N_y^{(1,1)}\right]\frac{\partial^2 w_1}{\partial y^2} + \left[N_{xy}^{(2)} + N_{xy}^{(1,1)}\right]\frac{\partial^2 w_1}{\partial x \partial y}$$

(3.21)

R.H.S. of the equation (3.21) is the sum of three terms

R.H.S = Term 1 + Term 2 + Term 3

Analyzing term by term

$$\text{Term 1 } = \frac{E_x h}{(1-\nu_x \nu_y)}\left[\left(\frac{\partial u_2}{\partial x} + \nu_y \frac{\partial v_2}{\partial y}\right) + \frac{1}{2}\left[\left(\frac{\partial w_1}{\partial x}\right)^2 + \nu_y \left(\frac{\partial w_1}{\partial y}\right)^2\right]\right]\frac{\partial^2 w_1}{\partial x^2}$$

substituting the value of $u_2, v_2$ & $w_1$ in Term 1 and simplifying we get co-efficient of $SinMxSinNy$ in Term1 as

$$C_1 = -\frac{E_x}{\left(1-\nu_x \nu_y\right)}\left[U_2 + 2M(A+B) + \nu_y\left[V_2 + 2N\left(C+D\right)\right]\right]M^2 W_1$$

Similarly co-efficient of $SinMxSinNy$ in Term 2

$$C_2 = -\frac{E_y}{\left(1-\nu_x \nu_y\right)}\left[V_2 + 2N\left(C+D\right) + \nu_x\left[U_2 + 2M(A+B)\right]\right]N^2 W_1$$

and co-efficient of $SinMxSinNy$ in Term 3

$$C_3 = MNG_{xy}h\left[8\left(BN + DN\right)W_1 - MNW_1^3\right]$$

Hence co-efficient of $SinMxSinNy$ in Term 2 in the R.H.S of equation

$$C = C_1 + C_2 + C_3$$

So the equation becomes

$$L.H.S = (C_1 + C_2 + C_3) SinMxSinNy + \text{ A term free of } SinMxSinNy$$

The homogeneous part of equation (3.21) is same as that in (3.9b). Hence $SinMxSinNy$ will also be a homogeneous solution to the equation (3.21). But a term of $SinMxSinNy$ appears on the right-hand-side. Therefore no solution to the equation (3.21) will be possible that satisfies the boundary conditions unless co-efficient of $SinMxSinNy$ on R.H.S =0

$$\therefore C_1 + C_2 + C_3 = 0$$

Substituting the values of coefficients and solving this gives the value of $W_1^2$ as

$$W_1^2 = \frac{8P^{(2)}}{bh} \frac{M^2}{\left[ \frac{(2-\nu_x\nu_y)}{(1-\nu_x\nu_y)} \left( E_x M^4 + E_y N^4 \right) - \frac{\left( E_x\nu_y + E_y\nu_x \right) M^2 N^2}{(1-\nu_x\nu_y)} \right]} \qquad (3.22)$$

## 3.4 Calculation of effective width

Axial shortening is given by

$$\Delta = u(0,y) - u(a,y)$$

$$u(x,y) = u_0(x,y) + u_2(x,y)$$

Now $u_0(0,y) - u_0(a,y) = U_0\left(-\frac{a}{2}\right) - U_0\left(\frac{a}{2}\right)$

$$= -U_0 a$$

$$u_2 = u_{2p} + u_{2h}$$

$$u_{2p}(0,y) - u_{2p}(a,y) = 0$$

$$u_{2h}(0,y) - u_{2h}(a,y) = -U_2 a$$

Hence the Total shortening

$$\Delta = u(0,y) - u(a,y)$$

$$\Delta = -U_0 a - U_2 a$$

Substituting the values of $U_0$ and $U_2$

$$\Delta = \frac{P^{(0)}}{E_x hb} a + \alpha^2 \left[ \frac{P^{(2)}}{E_x hb} + \frac{\left(M^2 - \nu_y N^2\right)}{\left(1 - \nu_x \nu_y\right)} \frac{W_1^2}{4} \right] a$$

$$\frac{\Delta}{a} = \frac{\left(P^{(0)} + \alpha^2 P^{(2)}\right)}{E_x hb} + \frac{\alpha^2 \left(M^2 - \nu_y N^2\right)}{\left(1 - \nu_x \nu_y\right)} \frac{W_1^2}{4}$$

$$\frac{\Delta}{a} = \frac{P}{E_x hb} + \frac{\alpha^2 \left(M^2 - \nu_y N^2\right)}{\left(1 - \nu_x \nu_y\right)} \frac{W_1^2}{4}$$

Effective width is given by $b_e = \frac{P}{E_x \Delta} \frac{a}{h}$

so

$$\frac{b_e}{b} = \frac{\frac{P}{E_x hb}}{\frac{P}{E_x hb} + \frac{\alpha^2 \left(M^2 - \nu_y N^2\right)}{\left(1 - \nu_x \nu_y\right)} \frac{W_1^2}{4}} \tag{3.23}$$

## 3.5 Non dimensionalization of expressions

The relations obtained in this chapter are non-dimensionalised in this section so that all the expressions can be evaluated using just the aspect ratio of plate and the ratio of properties in two orthogonal directions.

From equation (3.22)

$$W_1^{\ 2} = \frac{8P^{(2)}}{bh} \frac{M^2 \left(1 - \nu_x \nu_y\right)}{\left[\left(2 - \nu_x \nu_y\right)\left(E_x M^4 + E_y N^4\right) - \left(E_x \nu_y + E_y \nu_x\right) M^2 N^2\right]}$$

Let $\frac{E_y}{E_x} = \frac{\nu_y}{\nu_x} = c$ & $\frac{a}{b} = \lambda$ (3.24)

Substituting $M = \frac{m\pi}{a}$ & $N = \frac{n\pi}{b}$

The equation reduces to

$$W_1^{\,2} = \frac{8P^{(2)}}{\pi^2 h E_x} \frac{m^2\left(1 - c\nu_x^{\,2}\right)a\lambda}{\left[\left(2 - c\nu_x^{\,2}\right)\left(m^4 + cn^4\lambda^4\right) - 2c\nu_x m^2 n^2 \lambda^2\right]} \tag{3.25}$$

Let shortening at the start of buckling i.e. at $P = P^{(0)}$ be $\Delta = \Delta_0$

Then $\Delta_0 = u(0, y) - u(a, y)$

and $u = u_0 = U_0\left(x - \frac{a}{2}\right)$

therefore $\Delta_0 = u_0(0, y) - u_0(a, y)$

$$= U_0\left(-\frac{a}{2}\right) - U_0\left(\frac{a}{2}\right)$$

$$= -U_0 a$$

hence $\Delta_0 = \frac{P^{(0)}a}{E_x hb}$

or $\frac{\Delta_0}{a} = \frac{P^{(0)}}{E_x hb}$

Now from previous analysis

$$\frac{\Delta}{a} = \frac{P}{E_x hb} + \frac{\alpha^2\left(M^2 - \nu_y N^2\right)}{\left(1 - \nu_x \nu_y\right)} \frac{W_1^{\,2}}{4}$$

dividing the two equations

$$\frac{\Delta}{\Delta_0} = \frac{P}{P^{(0)}} + \frac{\alpha^2\left(M^2 - \nu_y N^2\right)}{\left(1 - \nu_x \nu_y\right)} \frac{W_1^{\,2}}{4} \frac{E_x hb}{P^{(0)}}$$

after substituting

$$\alpha^2 = \frac{P - P^{(0)}}{P^{(2)}} \qquad \& \qquad W_1^{\ 2} = \frac{8P^{(2)}}{\pi^2 h E_x} \frac{m^2\left(1 - c\nu_x^{\ 2}\right)a\lambda}{\left[\left(2 - c\nu_x^{\ 2}\right)\left(m^4 + cn^4\lambda^4\right) - 2c\nu_x m^2 n^2 \lambda^2\right]}$$

this reduces to

$$\frac{\Delta}{\Delta_0} = \frac{P}{P^{(0)}} + \frac{2m^2\left(m^2 - c\nu_x n^2 \lambda^2\right)}{\left[\left(2 - c\nu_x^{\ 2}\right)\left(m^4 + cn^4\lambda^4\right) - 2c\nu_x m^2 n^2 \lambda^2\right]} \frac{\left(P - P^{(0)}\right)}{P^{(0)}}$$

Now Let

$$\frac{2m^2\left(m^2 - c\nu_x n^2 \lambda^2\right)}{\left[\left(2 - c\nu_x^{\ 2}\right)\left(m^4 + cn^4\lambda^4\right) - 2c\nu_x m^2 n^2 \lambda^2\right]} = K \qquad (3.26)$$

which results in

$$\frac{\Delta}{\Delta_0} = \frac{P}{P^{(0)}} + K\frac{\left(P - P^{(0)}\right)}{P^{(0)}}$$

or

$$\frac{P}{P^{(0)}} = \frac{1}{1+K}\frac{\Delta}{\Delta_0} + \frac{K}{1+K}$$

and

$$\frac{b_e}{b} = \frac{\dfrac{P}{E_x h b}}{\dfrac{P}{E_x h b} + \dfrac{\alpha^2\left(M^2 - \nu_y N^2\right)}{\left(1 - \nu_x \nu_y\right)}\dfrac{W_1^{\ 2}}{4}}$$

which reduces to

$$\frac{b_e}{b} = \frac{\dfrac{P}{P_{cr}}}{(1+K)\dfrac{P}{P_{cr}} - K} \tag{3.27}$$

$$\because P^{(0)} = P_{cr}$$

## 3.6 Graphical analysis

### 3.6.1 Load deflection curve

First Graph is drawn between the $\dfrac{P}{P_{cr}}$ Vs $\dfrac{\Delta}{\Delta_0}$

Recalling the expression $\dfrac{\Delta}{\Delta_0} = \dfrac{P}{P^{(0)}} + K\dfrac{\left(P - P^{(0)}\right)}{P^{(0)}}$

Where $K = \dfrac{2m^2\left(m^2 - c\nu_x n^2\lambda^2\right)}{\left[\left(2 - c\nu_x{}^2\right)\left(m^4 + cn^4\lambda^4\right) - 2c\nu_x m^2 n^2\lambda^2\right]}$

Graph is plotted for m=1 and m=2 and other values fix as shown.

| m | n | $\nu_x$ | c | λ | *K* | η |
|---|---|---|---|---|---|---|
| 1 | 1 | 0.25 | 0.2 | 1 | 0.83151 | 0.2 |
| 2 | 1 | 0.25 | 0.5 | 2 | 0.647399 | 0.5 |

Where $\dfrac{E_y}{E_x} = \dfrac{\nu_y}{\nu_x} = c$ $\qquad \dfrac{a}{b} = \lambda$ and $\qquad \eta = \dfrac{a^4}{m^4}\dfrac{n^4}{b^4}\dfrac{E_y}{E_x}$

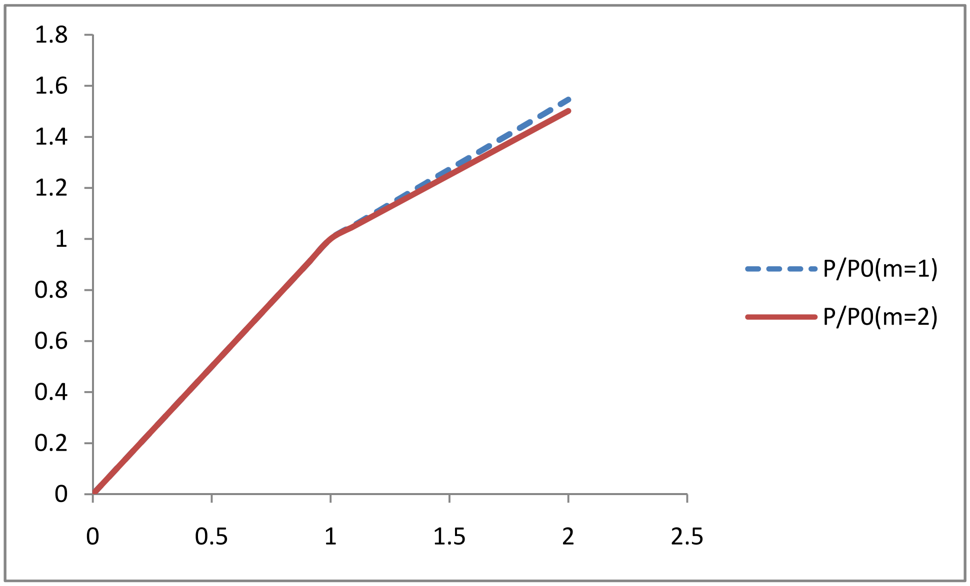


**Fig 3.2:** Load vs. Deflection curve for c=.2 and λ=1

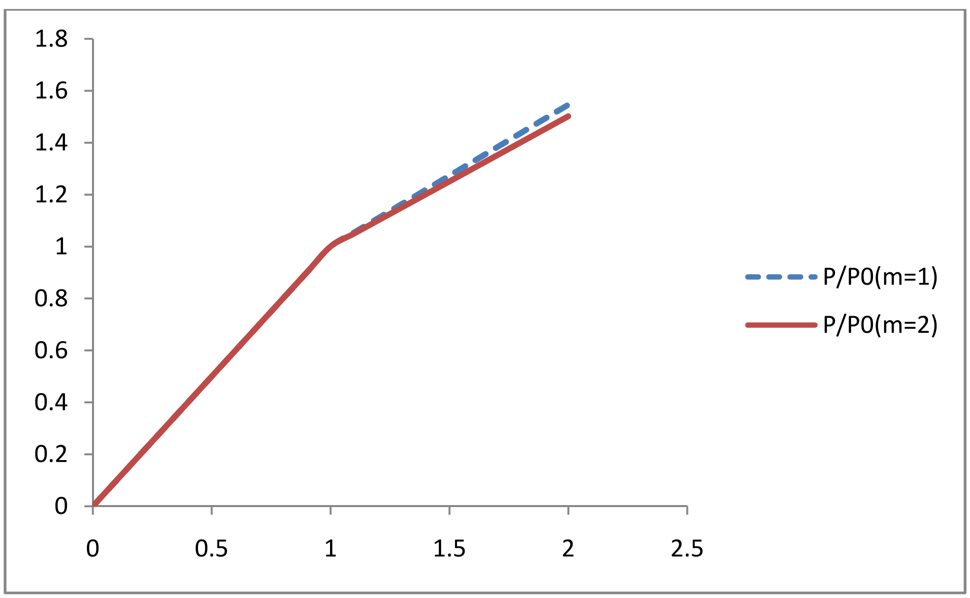


**Fig 3.3**: Load vs. Deflection curve for c=.5 and λ=1

As we can see, the graph $\frac{P}{P_{cr}}$ Vs $\frac{\Delta}{\Delta_0}$ is linear, which is expected since we haven't introduced any second order term of the deflection although we have taken non linearity in deformation into account. Now the curves have been shown for aspect ratio λ=1 that is a square plate.

### 3.6.2 Comparison of results of $\frac{b_e}{b}$ by two methods:

$\frac{b_e}{b}$ has been derived by two methods.

**3.6.2a) Successive approximation Method**

$$\frac{b_e}{b} = \frac{\frac{P}{P_{cr}}}{(1+K)\frac{P}{P_{cr}} - K} \qquad \text{where} \qquad K = \frac{2m^2\left(m^2 - c\nu_x n^2 \lambda^2\right)}{\left[\left(2 - c\nu_x{}^2\right)\left(m^4 + cn^4\lambda^4\right) - 2c\nu_x m^2 n^2 \lambda^2\right]}$$

**3.6.2b) Stress Method**

$$\frac{b_e}{b} = \frac{\frac{P}{P_{cr}}}{\frac{P}{P_{cr}} + \frac{2(\frac{P}{P_{cr}} - 1)}{(1+\eta)}} \qquad \text{where} \qquad \eta = \frac{a^4}{m^4}\frac{n^4}{b^4}\frac{E_y}{E_x}$$

Graph is plotted for m=1 and m=2 and other values fix as shown.

| m | n | $\nu_x$ | c | $\lambda$ | $K$ | $\eta$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 0.25 | 0.2 | 1 | 0.83151 | 0.2 |
| 1 | 1 | 0.25 | 0.5 | 1 | 0.647399 | 0.5 |

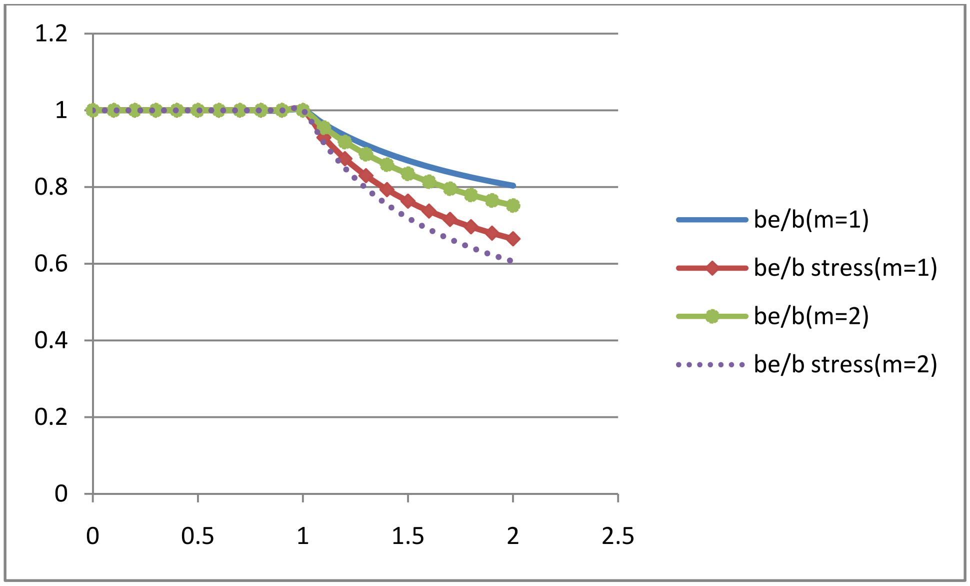


**Fig 3.4:** Effective width vs. deflection curve of orthotropic plate for c=0.2 and λ=1

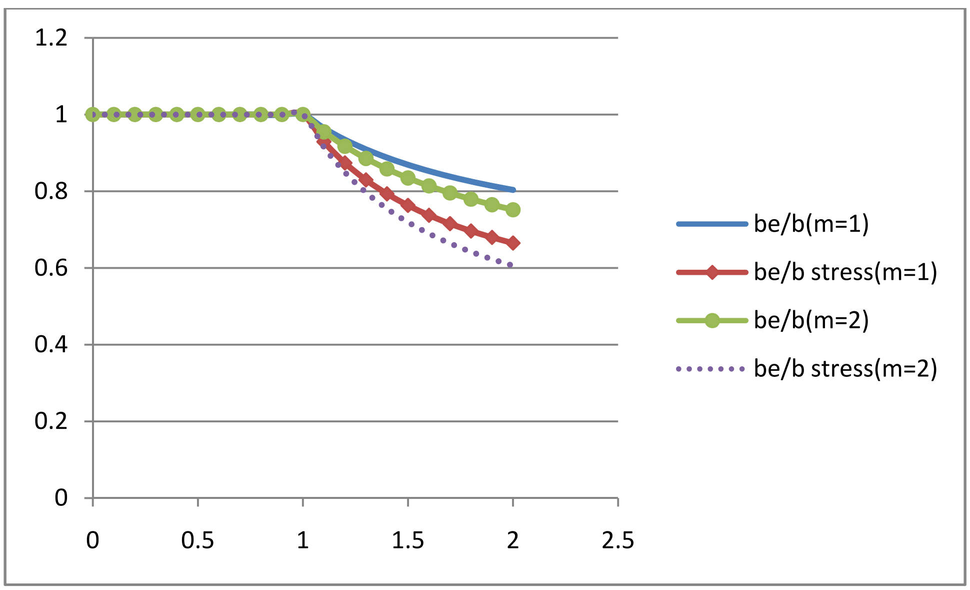


**Fig 3.5:** Effective width vs. deflection curve of orthotropic plate for c=0.5 and λ=1

## 3.7 Conclusion

From effective width vs. deflection curves of the two methods: 1) Stress Method and 2) successive approximation method, results have been compared and it can been seen that the values by two methods agree to an extent. Little deviation can be attributed to fact that the two methods are approximate methods. Also as the value of m increases or number of half sine curves in which the plate buckle increases, the effective width decreases.

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